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THE NECESSITY OF THE WIENER TEST FOR SOME SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

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1.Introduction

In this paper we will deal with the Wiener criterion on the regularity of a point of the boundary for subelliptic nonlinear problems. Let Ω be an open bounded set in R^N; we consider C^∞-vector fields X_i, $i = 1, \dots, m$, satisfying an Hörmander condition.

We recall that there is an intrinsic distance denoted by d, which is related to the vector fields, [10,11,14,17,18], and we write $B(x,r) = \{y|d(x,y) < r\}$, $B(r) = B(0,r)$, where 0 is the point $(0,0,\dots,0)$ in R^N.

Let $H^1(\Omega, X)(H_0^1(\Omega, X))$ be the completion of $C^\infty(\Omega)(C_0^\infty(\Omega))$ for the norm

$$|||v||| = (\sum_{i=1}^m \| X_i(v) \|_2^2 + \| v \|_2^2)^{\frac{1}{2}}$$

where $\| \,.\, \|_p$ denotes the usual L^p norm on Ω and $a_{ij}(x)$ be bounded measurable functions on Ω such that $a_{ij} = a_{ji}$ and

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2 \tag{1.1}$$

for every $\xi \in R^m$ and a.e. $x \in \Omega$. We consider the bilinear form on $H^1(\Omega, X)$ given by

$$a(u,v) = \int_\Omega \sum_{i,j=1}^m a_{ij} X_i u X_j v dx$$

and a function $f(x,z,q)$ on $\Omega \times R \times R^m$ continuous in (z,q) for each fixed x and measurable in x for each fixed (z,q) such that

$$|f(x,z,q)| \leqslant a + b|q|^2 \tag{1.2}$$

for $q \in R^N$, $z \in R$ and for a.e. $x \in \Omega$.

We consider the problem

$$-\sum_{i,j=1}^m X_j(a_{ij}X_i u) = f(x,u,Xu) \text{ in } \Omega \tag{1.3}$$

where $Xu = (X_i u,\ i = 1,\dots,m)$. We say that $u \in H^1(R^N, X) \cap L^\infty(R^N)$ is a *bounded weak solution* to (1.3) with boundary value $\Phi \in H^1(R^N, X) \cap L^\infty(R^N)$ if

(i) $u - \Phi \in H_0^1(\Omega, X)$, and

(ii) for all $\psi \in H_0^1(\Omega, X) \cap L^\infty(\Omega)$

$$a(u,\psi) = \int_\Omega f(x,u,Xu)\psi dx.$$

There is a natural notion of *capacity* associated to our problem; namely let K be a compact set in the open set O then the capacity $cap(K, O)$ is defined as the infimum of the integral

$$\int_O \sum_{i,j}^{m} a_{ij} X_i v X_j v dx$$

taken on all functions $v \in C_0^1(O)$ such that $v \geqslant 1$ on K. The definition is extended to open sets $A \subseteq O$ by

$$\mathrm{cap}(A, O) = \sup\{\mathrm{cap}(K, O); K \subseteq A \text{ compact}\}$$

and to arbitrary subsets E in O by

$$\mathrm{cap}(E, O) = \inf\{\mathrm{cap}(A, O); E \subseteq A, A \subseteq O \text{ open set}\}$$

The notion of *potential* and *capacitary measure* of a set are analogous to the one given for the usual newtonian capacity and have the same properties.

We say that a set E has *capacity zero* if $cap(E, O) = 0$ for one (hence for all) open sets O such that $\overline{E} \subseteq O$. If a property depending on $x \in O$, holds except a subset of capacity zero, then we say that this property holds *quasi - everywhere (q.e.)*.

The notion of capacity implies in a natural way a notion of *quasi - continuity* and we have that a function $v \in H^1(\Omega, X)$ has a quasi-continuous representative $\tilde{v}$.

For all properties connected with the notion of capacity we refer to the book of Fukushima,[12].

Consider now the function Φ and let x_0 be a point in $\partial\Omega$ we say that Φ is continuous at x_0 with respect to $\partial\Omega$ if $cap(B(x_0, R) \cap \partial\Omega) > 0$ for $R > 0$ and $q.e$ - $osc_{B(x_0,r)\cap\partial\Omega}\Phi$ converges to 0 as $r \to 0$. In this case we can take as value of Φ at x_0 the q.e.-limit of Φ when $x \in \partial\Omega$ approaches x_0.

We say that x_0 is a *regular (local) point* for $\partial\Omega$ with respect to (1.3) if for every open set $\Omega' \subseteq \Omega$ with $x_0 \in \partial\Omega' \cap \partial\Omega$ every bounded weak solution u of (1.3) with boundary data $\Phi \in H^1(R^N, X) \cap L^\infty(R^N)$ on $\partial\Omega'$ continuous at x_0 is such that

$$\lim_{x \to x_0, x \in \Omega'} u = \Phi(x_0).$$

We recall that there is a well known criterion, namely the *Wiener criterion*, for the regularity of x_0 in the case $f = 0$, [15].
We now recall the above criterion.

Theorem 1.1. *Let $f = 0$ and*

$$\delta(r) = \frac{cap(\Omega^c \cap B(x, r), B(x, 2r))}{cap(B(x, r), B(x, 2r))}.$$

Then a necessary and sufficient condition for the regularity of x_0 with respect to $\partial\Omega$ is that

$$\int_0^{R_0} \delta(r) \frac{dr}{r} = +\infty \tag{1.4}$$

where R_0 is positive and fixed.

Theorem 1.1 can be proved adapting the methods of [2] following the techniques of [3,5]; a more general result will be proved, in a following paper, [6], in the framework of Dirichlet forms. Using exponential test functions of the type choosen in [7,13], we can extend the sufficient part of the criterion to the case $f \neq 0$.

In this paper we are concerned with the necessary part of the criterion and we will prove the following result:

Theorem 1.2. *The condition (1.4) is necessary for the regularity with respect to Ω of the point x_0.*

In the following section we reduce the nonlinear problem to a linear one and we give some results on the existence of the Green function of such a problem; in section 3 we finally prove the Theorem 1.2. We recall that this problem has been studied in the case of unifomly elliptic operators by Adams and Heard,[1], and in the monotone case with nonlinear principal part by Skrypnik, [19]. (this Author considers general nonlinear elliptic monotone problems); those authors prove the result of Theorem 1.2 under the additional assumption of Dini-continuity for the coefficients a_{ij}. Nothing seems to be known in the nonlinear case with Hörmander's vector fields.

Finally we observe that the methods given here works again in the presence of a weight ω in the A_2 Muckenhoupt's class (with respect to the intrinsic balls, see [9] for the definition in the euclidean framework). In this case we have to take care of the possible presence of points of positive capacity (see [2][6] for the analogous linear elliptic case). For the tool used in the weighted case as weighted Poincare and Sobolev inequalities we refer to [16].

2. Reduction to a linear case.

The first proposition we will prove show that there exists a solution to our problem connected with a suitable super- and sub-solution to a linear problem.

In all this section we consider an open set O such that, denoted by $\lambda_1(O)$ the first eigenvalue of the Dirichlet problem with zero boundary data of the operator $-\sum_{i=1}^m X_i^2$, we have

$$\frac{ab}{\lambda^2} < \lambda_1(O).$$

Proposition 2.1. *Let u be a bounded weak solution of*

$$-\sum_{i,j=1}^{m} X_j(a_{ij}X_i u) = f(x,u,Xu) \tag{2.1}$$

a.e. in O with boundary data Φ. Then the functions $exp(\pm\frac{b}{\lambda}u)$ are subsolutions of the problem

$$\sum_{i,j=1}^{m}\int_O a_{ij}X_iV^{\pm}X_jv\,dx - \int_O \frac{ab}{\lambda}V^{\pm}v\,dx = 0 \;\; \forall v \in H_0^1(O,X) \tag{2.2}$$

with boundary values $exp(\pm\frac{b}{\lambda}\Phi)$.

Proof: The proof follows by easy computations. In fact for $v \in H_0^1(O,X) \cap L^\infty(O)$ with $v \geqslant 0$ we have

$$\sum_{i,j=1}^{m}\int_O a_{ij}X_i(\exp(\pm\frac{b}{\lambda}u)X_jv\,dx = \pm\frac{b}{\lambda}\sum_{i,j=1}^{m}\int_O a_{ij}(\exp(\pm\frac{b}{\lambda}u)X_iuX_jv\,dx =$$

$$\pm\frac{b}{\lambda}\sum_{i,j=1}^{m}\int_O a_{ij}X_iuX_j(\exp(\pm\frac{b}{\lambda}u)v)dx -$$

$$(\frac{b}{\lambda})^2\sum_{i,j=1}^{m}\int_O \exp(\pm\frac{b}{\lambda}u)va_{ij}X_iuX_ju\,dx \leqslant$$

$$\frac{ab}{\lambda}\int_O(\exp(\pm\frac{b}{\lambda}u)v)dx + \int_O |Xu|^2(\exp(\pm\frac{b}{\lambda}u)v)dx -$$

$$(\frac{b}{\lambda})^2\sum_{i,j=1}^{m}\int_O \exp(\pm\frac{b}{\lambda}u)va_{ij}X_iuX_ju\,dx \leqslant$$

$$\frac{ab}{\lambda}\int_O(\exp(\pm\frac{b}{\lambda}u)v)dx.$$

Denote now by $V^{\pm}$ the solutions of (2.2) with boundary value $\exp(\pm\frac{b}{\lambda}\Phi)$ and define $u^{\pm} = \pm\frac{b}{\lambda}\log V^{\pm}$; then if u is a bounded weak solution of (2.1) with boundary data Φ we have

$$u^{-} \leqslant u \leqslant u^{+}. \tag{2.3}$$

Proposition 2.2. *The problem (2.1) with boundary data Φ has at least one solution u with*

$$u^{-} \leqslant u \leqslant u^{+}.$$

Sketch of the Proof: We only sketch the proof:
(1) We regularize f by

$$f_\varepsilon = \frac{f}{1+\varepsilon f}$$

and we denote by (P) and (P_ε) the boundary problems relative to f and f_ε. We observe that

$$|f_\varepsilon(x,z,q)| \leqslant |f(x,z,q)|$$

where $\varepsilon = \frac{1}{n}$.

Denote by u_ε the solutions, whose existence can be easily proved, of (P_ε). The existence of a bounded weak solution of problem (P_ε) can be proved for example by a fixed point method in $H^1_{\mathrm{loc}}(O,X)$ taking also in account the local estimates for the linear problem given in [5] and that the proof of the global L^∞ estimate for the linear problem given in [5] in the case of balls hold again for general bounded open set O. Then from (2.3) we obtain

$$u^{-} \leqslant u_\varepsilon \leqslant u^{+}. \tag{2.4}$$

From (2.4) we have easily that u_ε is bounded uniformly with respect to ε.
(2) It follows easily from that u_ε is bounded in $H^1(O,X)$ and from [13] we have that u_ε is bounded in C^α locally in O.
(3) At least after extraction of a subsequence we have that u_ε - Φ converges weakly in $H^1_0(O,X)$ and strongly in $L^p(O)$ for every finity $p \geqslant 1$ to u–Φ. From the C^α_{loc} estimate we can also suppose that u_ε converges to u unifomly locally in O; then easy computations proves that u_ε converges to u strongly in $H^1_{\mathrm{loc}}(O,X)$.

The convergences in (3) prove that u is a solution of (1.3) in O with boundary data Φ.

Using the methods in [5] with the adaptations given for the usual unifomly elliptic case in [9] we obtain that

Proposition 2.3 *For every x in O there exists a Green function for (2.2) with singularity at $x \in O$ denoted by G^x_O. Moreover choosing $O = B(x,R)$ we have*

$$G^x_{B(x,R)} \approx \int_r^R \frac{\rho^2}{m(B(x,\rho))}\frac{d\rho}{\rho} \quad on\ \partial B(x,r)$$

for $r \leqslant \frac{R}{2}$. Moreover if we denote by $G^x_{\rho,B(x,R)}$ the regularized Green function (the definition is analogous to the one given in [5] in the case $ab = 0$) we have also

$$G^x_{\rho,B(x,R)} \approx \int_r^R \frac{\rho^2}{m(B(x,\rho))}\frac{d\rho}{\rho} \quad on\ \partial B(x,r)$$

for $2\rho < r < R$, and

$$\lim_{\rho\to 0} G^x_{\rho,B(x,R)} = G^x_{B(x,R)}$$

in $C^\alpha_{\mathrm{loc}}(B(x,R)\setminus\{x\}) \cap H^1_{\mathrm{loc}}(B(x,R)\setminus\{x\},X)$.

Taking into account Propositions 2.1 and 2.2 our result will be proved if, in the case of convergence of the Wiener integral, we construct a solution V^{+} of (2.2) relative to a boundary data $\Psi \in H^1(R^N, X) \cap L^\infty(R^N)$, $\Psi > \varepsilon > 0$ and to the set $\Omega_r = \Omega \cap B(x_0, r)$ with

$$\lim_{x \to x_0, x \in \partial\Omega_r} \Psi = 1 = \Psi(x_0)$$

such that

$$\liminf_{x \to x_0, x \in \Omega_r} V^+ < 1.$$

In fact if we consider the solution u of (2.1) in Ω_r with boundary data $\Phi = \frac{\lambda}{b} \log \Psi \in H^1(R^N, X) \cap L^\infty(R^N)$ then

$$\liminf_{x \to x_0, x \in \Omega_r} u \leqslant \liminf_{x \to x_0, x \in \Omega_r} u^+ < 0 = \Phi(x_0)$$

where $u^+ = \frac{\lambda}{b} \log V^+$.

3.Proof of Theorem 1.2.

We denote by $H^{-1}(O, X)$ the dual space of $H_0^1(O, X)$. Moreover in this section all the potentials and the Green functions are taken with respect to the form in (2.2) and we can assume (without loss of generality) $x_0 = 0$. We recall that we assume again $\frac{ab}{\lambda^2} < \lambda_1(O)$.

Proposition 3.1. *Let μ be a bounded positive measure in $H^{-1}(O; X), O = B(2R)$, with support in $\overline{B(R)}$.*

Let v_R be the potential of μ in $B(2R)$. Assume

$$\int_0^{2R} \mu(B(\rho)) \frac{\rho^2}{m(B(\rho))} \frac{d\rho}{\rho} < +\infty. \tag{3.1}$$

Denote G the Green functions with singularity at 0 with respect to $B(2R)$; then $G(x, 0)$ is integrable with respect to the measure μ and the value

$$\hat{v}_R(0) = \int_{B(2R)} G(x, 0) \mu(dx)$$

is well defined.

Moreover the limit

$$v_R(0) = \lim_{\rho \to 0} \frac{1}{m(B(\rho))} \int_{B(\rho)} v_R(x) dx$$

exists finite and

$$v_R(0) = \hat{v}_R(0) \leqslant C \int_0^{2R} \mu(B(\rho)) \frac{\rho^2}{m(B(\rho))} \frac{d\rho}{\rho}. \tag{3.2}$$

Proof: Let us prove the Lemma for a fixed $R = \overline{R}$. We observe that

$$\mu(B(\rho)) \left(\int_\rho^{2\overline{R}} \frac{s^2}{m(B(s))} \frac{ds}{s} \right) \leqslant \int_\rho^{2\overline{R}} \mu(B(s)) \frac{s^2}{m(B(s))} \frac{ds}{s} \tag{3.3}$$

hence

$$\liminf_{r \to 0} \overline{\mu(B(r))} \left(\int_r^{2\overline{R}} \frac{\rho^2}{m(B(\rho))} \frac{d\rho}{\rho} \right) < +\infty.$$

Integating by parts and taking the size of G into account we obtain for arbitrary $0 < r < R < \overline{R}$

$$\int_{r<d(x,0)<R} G(x,0)\mu(dx) \approx \int_{r<d(x,0)<R} (\int_{d(x,0)}^{2\overline{R}} \frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho})\mu(dx) =$$

$$\int_r^R \mu(B(\rho))\frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho} - \mu(\overline{B(R)}) \int_R^{2\overline{R}} \frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho} + \tag{3.4}$$

$$\mu(\overline{B(r)}) \int_r^{2\overline{R}} \frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho}$$

Putting $\sigma = r$ and letting $R \to \overline{R}$ we have

$$\int_{\sigma<d(x,0)<\overline{R}} G(x,0)\mu(dx) \leqslant$$

$$C[\int_0^{2\overline{R}} \mu(B(\rho))\frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho} + \mu(\overline{B(\sigma)}) \int_\sigma^{2\overline{R}} \frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho}] \leqslant \tag{3.5}$$

$$2C\int_0^{2\overline{R}} \mu(B(\rho))\frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho}.$$

We now let $\sigma \to 0$ and we obtain that $G(x,0)$ is indtegrable with respect to μ and

$$\int_{B(2\overline{R})} G(x,0)\mu(dx) \leqslant C\int_0^{2\overline{R}} \mu(B(\rho))\frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho}. \tag{3.6}$$

(we denote by C possibly different constants depending on X_i, $i = 1,\dots,m$, and on $\frac{\lambda}{\Lambda}$). From (3.6) we have

$$\lim_{\sigma\to 0}\int_{B(\sigma)} G(x,0)\mu(dx) = 0. \tag{3.7}$$

Now we prove that $\hat{v}_{\overline{R}}(0) = v_{\overline{R}}(0)$. By the estimates on the regularized Green function G_ρ we obtain that

$$G_\rho(x,0) \leqslant G(x,0).$$

We recall that

$$\lim_{\rho\to 0} G_\rho(x,0) = G(x,0) \tag{3.8}$$

everywhere for $x \neq 0$ and uniformly for $d(x,0) > \sigma$.
Being $G(x,0)$ integrable on $B(2\overline{R})$, we have that for $\sigma \leqslant \frac{\overline{R}}{4}$

$$lim_{\rho\to 0}\int_{B(\sigma)} G_\rho(x,0)\mu(dx) = \int_{B(\sigma)} G(x,0)\mu(dx) \tag{3.9}$$

From (3.8) we have that

$$\lim_{\rho\to 0}\int_{d(x,0)>\sigma} G_\rho(x,0)\mu(dx) = \int_{d(x,0)>\sigma} G(x,0)\mu(dx) \tag{3.10}$$

From (3.9) and (3.10) it follows

$$v_{\overline{R}}(0) = \frac{1}{m(B(\rho))}\int_{B(\rho)} v_{\overline{R}}dx =$$

$$\lim_{\rho\to 0}\int_{B(2\overline{R})} G_\rho(x,0)\mu(dx) = \int_{B(2\overline{R})} G(x,0)\mu(dx) = \hat{v}_{\overline{R}}(0).$$

Proposition 3.2. *Let E_ρ, $\rho > 0$, be subsets of R^N such that*

$$E_r \cap B(\rho) \subseteq E_\rho \subseteq B(\rho) \subseteq B(r) \subseteq O$$

for every $0 < \rho < r$. Let μ_ρ be the capacitary measure of E_ρ in $B(2\rho)$; than for every $r > 0$ and $0 < \rho < r$ we have

$$\mu_r(B(\rho)) \leqslant \mu_\rho(\overline{B(\rho)})$$

Proof: Let w_ρ be the potential of E_ρ in $B(2\rho)$. We have

$$\sum_{i,j=1}^{m} \int_O a_{ij} X_i w_\rho X_j w_\rho dx \geqslant \sum_{i,j=1}^{m} \int_O a_{ij} X_i w_\rho X_j w_r dx$$

where $0 < \rho < r$.

The result follows easily from the above relation using the capacitary measures.

Now we choose $O = \Omega \cap B(2r) = \Omega_{2r}$, then for $0 < r < R$, R suitable, we have $\frac{ab}{\lambda^2} < \lambda_1(O)$, so we can use all the previous results.

Proof of Theorem 1.2: Let us suppose that

$$\int_O^R \delta(\rho)\frac{d\rho}{\rho} < +\infty \tag{3.11}$$

To prove Theorem 1.2 it is enough to prove that for r suitable with $0 < r < R$ there exists w_r solution of (2.2) in Ω_{2r} with boundary data $\Psi \in H^1(R^N, X) \cap L^\infty(R^N)$, $\Psi \geqslant \varepsilon > 0$, such that we have

$$\text{liminf}_{x\to 0} w_r(x) < 1. \tag{3.12}$$

The maximum principle show that to find w_r it is enough to prove that denoted by v_r the potential of $\Omega^c \cap B(r)$ in $B(2r)$ we have

$$\text{liminf}_{x\to 0} v_r(x) < 1. \tag{3.12'}$$

To prove (3.12′) it is enough to prove that for r suitable with $0 < r < R$ we have

$$\lim_{\rho\to 0} \frac{1}{m(B(\rho))}\int_{B(\rho)} v_r(x)dx = v_r(0) < 1. \tag{3.13}$$

Let μ_r be the capacitary measure of Ω_r with respect to $B(2r)$.

For every $r > 0$ we have $\text{supp}(\mu_r) \subseteq \overline{B(r)}$, then from (3.11) and from the Proposition 3.2 we obtain

$$\int_0^{2r} \mu(B(\rho))\frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho} < +\infty.$$

By Proposition 3.1 with $\mu = \mu_r$ we obtain

$$v_r(0) \leqslant C\int_0^{2r} \mu_r(B(\rho))\frac{\rho^2}{m(B(\rho))}\frac{d\rho}{\rho}.$$

Then from Proposition 3.2 we have also

$$v_r(0) \leqslant C\int_0^{2r} \delta(\rho)\frac{d\rho}{\rho}.$$

By letting $r \to 0$, we obtain from (3.11)

$$\lim_{r\to 0} v_r(0) = 0. \tag{3.14}$$

From (3.14) the relation (3.13) easily follows.

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